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INTERVAL GENERALIZATION OF THE BAYESIAN MODEL OF COLLECTIVE DECISION-MAKING IN CONFLICT SITUATIONS

O. A. Zhukovskaya[a] and L. S. Fainzil'berg[b] UDC 658.012.011

A constructive interval model of making a collective decision by an independent group of experts is developed. The model is based on a priori information about the frequency of experts' errors in estimating a random state of an object using a finite sample.

Keywords: *interval model, collective decision, conflict situation.*

Methods of collective decision-making are widely used in various fields [1–3]. Different approaches to constructing models of collective decisions are known, in particular, voting [4], ranking [5], averaging partial decisions of experts [6], algorithms of fuzzy rules [7], etc.

Bayesian models of collective decision-making by a group of independent experts under conflict conditions are proposed in [8]. However, these models employ a priori knowledge of the probabilities of experts' errors, which are often unknown in practice.

The development of interval analysis methods [9, 10] makes it possible to pass from point models to interval models based on Bayesian inference mechanisms [11, 12]. The purpose of the present paper is to develop (in the context of interval analysis) Bayesian models of collective decision-making in conflict situations, based on a priori knowledge of the frequency of errors made by experts using finite experimental samples.

PROBLEM STATEMENT

Let a plant be in one of two possible states: V_1 or V_2, randomly passing from one state to the other with a priori probabilities $P(V_1)$and $P(V_2)=1-P(V_1)$. Two experts, A_1 and A_2, using independent data, make a decision δ_i on the current state of the plant in the form of indicator functions:

$$\delta_i = k \text{ if } A_i \text{ decides for } V_k,\ i=1,2,\ k=1,2. \tag{1}$$

It is clear that the set of possible situations consist of four combinations of partial decisions (1). These decisions are agreed only in two cases (when the experts make the same decision), and in the other two cases the decisions are contradictory:

$$S_{12}:(\delta_1=1)\wedge(\delta_2=2);$$

$$S_{21}:(\delta_1=2)\wedge(\delta_2=1). \tag{2}$$

Let experts' error rates be estimated using a representative sample of *n* observations with known states of the plant:

$$P^*_{A_i}=\frac{E_{A_i}}{n},\quad i=1,2, \tag{3}$$

where E_{A_i} is the number of cases where the *i*th expert has made an incorrect decision.

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[a]National Technical University "Kiev Polytechnical Institute," Kiev, Ukraine, *fainzilberg@svitonline.com*. [b]International Scientific and Training Center of Information Technologies and Systems, National Academy of Sciences of Ukraine and Ministry of Education and Science of Ukraine, Kiev, Ukraine, *fainzilberg@svitonline.com*. Translated from Kibernetika i Sistemnyi Analiz, No. 3, pp. 133-144, May-June 2005. Original article November 1, 2004.

Based on a priori information available, it is necessary to construct a formal model minimizing the average probability of collective decision $D = D(\delta_1, \delta_2)$ on unknown states of the plant in conflict situations (2).

POINT MODEL OF COLLECTIVE DECISION

Let us first consider a formal model of collective decision-making $D = D(\delta_1, \delta_2)$ on the assumption that the probabilities of P_{A_i} experts' errors rather than the error rates (3) characterize expert' "qualifications." According to [8], under such assumptions, the conflict settling conditions can be formulated as follows.

LEMMA 1. For a collective decision $D = D(\delta_1, \delta_2)$ to ensure the minimum average probability of error for the known values of $P(V_k)$, $k = 1, 2$, and P_{A_i}, $i = 1, 2$, it is necessary and sufficient that, in a conflict situation S_{12}, a decision be made in favor of V_1 if

$$P_{A_2}(1 - P_{A_1}) > \lambda P_{A_1}(1 - P_{A_2}), \tag{4}$$

and in favor of V_2 if

$$P_{A_2}(1 - P_{A_1}) < \lambda P_{A_1}(1 - P_{A_2}), \tag{5}$$

and in a conflict situation S_{21}, a decision be made in favor of V_1 if

$$P_{A_1}(1 - P_{A_2}) > \lambda P_{A_2}(1 - P_{A_1}), \tag{6}$$

and in favor of V_2 if

$$P_{A_1}(1 - P_{A_2}) < \lambda P_{A_2}(1 - P_{A_1}), \tag{7}$$

where

$$\lambda = \frac{P(V_2)}{P(V_1)}.$$

Proof. According to statistical decision theory, the average error probability for the collective decision $D = D(\delta_1, \delta_2)$ is minimum if decisions on the maximum of posterior probabilities are made in all possible situations. For example, in the conflict situation S_{12}, make a decision in favor of V_1 if

$$P(V_1 / S_{12}) > P(V_2 / S_{12}), \tag{8}$$

and in favor of V_2 otherwise.

By the Bayesian formula, we have

$$P(V_1 / S_{12}) = \frac{P(V_1)P(S_{12} / V_1)}{P(S_{12})}, \qquad P(V_2 / S_{12}) = \frac{P(V_2)P(S_{12} / V_2)}{P(S_{12})}.$$

It is obvious that inequality (8) holds iff

$$P(V_1)P(S_{12} / V_1) > P(V_2)P(S_{12} / V_2). \tag{9}$$

By definition, $P(S_{12} / V_1)$ is the conditional probability that when the plant is in the state V_1, the expert A_1 has made a correct decision and the expert A_2 was mistaken. Since the experts' decisions are assumed independent, the product formula for probabilities yields

$$P(S_{12} / V_1) = (1 - P_{A_1})P_{A_2}. \tag{10}$$

Similarly,

$$P(S_{12} / V_2) = (1 - P_{A_2})P_{A_1}. \tag{11}$$

Then (9), (10), and (11) yield that in the conflict situation S_{12} a collective decision that minimizes the average error probability should be made according to conditions (4) and (5). We can similarly show that in the conflict situation S_{21} a collective decision minimizing average error probability should be made according to conditions (6) and (7). Lemma 1 is proved.

It is clear that the exact values of the probabilities P_{A_i}, $i=1,2$, appearing in the optimal model (4)–(7) are frequently unknown in practice. It is also clear that the probabilities P_{A_i} can be replaced with the point estimates (3) only for a sufficiently large number of observations *n*. Therefore, of practical interest is to generalize models (4)–(7) to the case where the confidence intervals I_{A_i} are used instead of the point values of the probabilities P_{A_i}.

According to [13, p. 332], the frequency P_X^* of a random event *X*, calculated from a sample of size *n*, determines, with confidence probability *β*, the confidence interval I_X for the probability P_X; we can write it in the center–radius form:

$$I_X = \langle P_X^c, r_X \rangle, \tag{12}$$

where

$$P_X^c = \frac{P_X^* + t_\beta^2 / 2n}{1 + t_\beta^2 / n}, \tag{13}$$

$$r_X = \frac{t_\beta \sqrt{\dfrac{P_X^*(1-P_X^*)}{n} + \dfrac{t_\beta^2}{4n^2}}}{1 + t_\beta^2 / n}. \tag{14}$$

In these relations,

$$t_\beta = \arg \Phi^*\left(\frac{1+\beta}{2}\right),$$

where $\Phi^*(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{r^2}{2}} d\tau$ is the normal distribution function.

Let us introduce the notation

$$L_1 = P_{A_2}(1 - P_{A_1}), \tag{15}$$

$$W_1 = \lambda P_{A_1}(1 - P_{A_2}), \tag{16}$$

$$L_2 = P_{A_1}(1 - P_{A_2}), \tag{17}$$

$$W_2 = \lambda P_{A_2}(1 - P_{A_1}) \tag{18}$$

and pass from the point values (15)–(18) appearing in models (4)–(7) to their interval analogs

$$\mathbf{L}_1 = I_{A_2}(1 - I_{A_1}), \tag{19}$$

$$\mathbf{W}_1 = \lambda I_{A_1}(1 - I_{A_2}), \tag{20}$$

$$\mathbf{L}_2 = I_{A_1}(1 - I_{A_2}), \tag{21}$$

$$\mathbf{W}_2 = \lambda I_{A_2}(1 - I_{A_1}), \tag{22}$$

where I_{A_1} and I_{A_2} are the confidence intervals (12) of the appropriate probabilities.

Applying arithmetic operations for the intervals in center–radius form [14], we represent intervals (19) and (20) as

$$\mathbf{L}_1 = \langle L_1^c, r_{L_1} \rangle = \langle P_{A_2}^c, r_{A_2} \rangle (1 - \langle P_{A_1}^c, r_{A_1} \rangle) = \langle P_{A_2}^c, r_{A_2} \rangle \langle 1 - P_{A_1}^c, r_{A_1} \rangle$$

$$= \langle P_{A_2}^c (1 - P_{A_1}^c) + r_{A_1} r_{A_2}, P_{A_2}^c r_{A_1} + r_{A_2}(1 - P_{A_1}^c) \rangle\rangle, \tag{23}$$

$$\mathbf{W}_1 = \langle W_1^c, r_{W_1}\rangle = \lambda\,\langle P_{A_1}^c, r_{A_1}\rangle (1-\langle P_{A_2}^c, r_{A_2}\rangle) = \lambda\,\langle P_{A_1}^c, r_{A_1}\rangle\,\langle 1-P_{A_2}^c, r_{A_2}\rangle$$

$$= \lambda\,\langle P_{A_1}^c (1-P_{A_2}^c) + r_{A_1} r_{A_2},\, P_{A_1}^c r_{A_2} + r_{A_1}(1-P_{A_2}^c)\rangle. \qquad (24)$$

Similarly, we can represent intervals (21) and (22) as

$$\mathbf{L}_2 = \langle L_2^c, r_{L_2}\rangle = \langle P_{A_1}^c (1-P_{A_2}^c) + r_{A_1} r_{A_2},\, (1-P_{A_2}^c) r_{A_1} + P_{A_1}^c r_{A_2}\rangle, \qquad (25)$$

$$\mathbf{W}_2 = \langle W_2^c, r_{W_2}\rangle = \lambda\,\langle P_{A_2}^c (1-P_{A_1}^c) + r_{A_1} r_{A_2},\, (1-P_{A_1}^c) r_{A_2} + P_{A_2}^c r_{A_1}\rangle. \qquad (26)$$

According to conditions (4)–(7) with (15)–(18), an optimal collective decision should be based on a comparison of the point quantities L_i and W_i. It is obvious that any value of $L_i \in \mathbf{L}_i$ will be greater (or smaller) than any value of $W_i \in \mathbf{W}_i$ if the intervals $\mathbf{L}_i = \langle L_i^c, r_{L_i}\rangle$ and $\mathbf{W}_i = \langle W_i^c, r_{W_i}\rangle$ do not intersect, i.e.,

$$L_i^c - r_{L_i} > W_i^c + r_{W_i} \qquad (27)$$

or

$$L_i^c + r_{L_i} < W_i^c - r_{W_i}. \qquad (28)$$

In the conflict situation S_{12}, we can represent condition (27) with (23) and (24) as

$$P_{A_2}^c (1-P_{A_1}^c) + r_{A_1} r_{A_2} - P_{A_2}^c r_{A_1} - r_{A_2}(1-P_{A_1}^c)$$

$$> \lambda\left(P_{A_1}^c (1-P_{A_2}^c) + r_{A_1} r_{A_2} + P_{A_1}^c r_{A_2} + r_{A_1}(1-P_{A_2}^c)\right),$$

whence elementary transformations yield

$$(P_{A_2}^c - r_{A_2})(1-P_{A_1}^c - r_{A_1}) > \lambda\,(P_{A_1}^c + r_{A_1})(1-P_{A_2}^c + r_{A_2}), \qquad (29)$$

and we can write condition (28) with (23) and (24) as

$$(P_{A_2}^c + r_{A_2})(1-P_{A_1}^c + r_{A_1}) < \lambda\,(P_{A_1}^c - r_{A_1})(1-P_{A_2}^c - r_{A_2}). \qquad (30)$$

Similarly, in the conflict situation S_{21} conditions (27) and (28) become

$$(P_{A_1}^c - r_{A_1})(1-P_{A_2}^c - r_{A_2}) > \lambda\,(P_{A_2}^c + r_{A_2})(1-P_{A_1}^c + r_{A_1}), \qquad (31)$$

$$(P_{A_1}^c + r_{A_1})(1-P_{A_2}^c + r_{A_2}) < \lambda\,(P_{A_2}^c - r_{A_2})(1-P_{A_1}^c - r_{A_1}). \qquad (32)$$

Let us show that conditions (29)–(32) allow constructing a formal model of optimal decisions in conflict situations (2) when the exact values of the probabilities P_{A_i} of experts' errors are unknown, $i = 1, 2$, and only information on the error rate (3) of each expert on a finite sample of observations is available.

THEOREM 1. Let two experts, A_1 and A_2, irrespective of each other, make decisions as to the current state of the plant $V_j \in \{V_1, V_2\}$ with a priori probabilities $P(V_1)$ make, $P(V_2) = 1 - P(V_1)$, error rates $P_{A_i}^*(n)$ for each expert being preestimated using a sample of n observations. Then, with the probability β, the collective decision $D = D(\delta_1, \delta_2)$ minimizes the average error probability if:

1) in the conflict situation S_{12}, a decision is made in favor of V_1 when condition (29) holds and in favor of V_2 when condition (30) holds; and

2) in the conflict situation S_{21}, a decision is made in favor of V_1 when condition (31) holds and in favor of V_2 when condition (32) holds.

Proof. Let us first show that when condition (29) holds, condition (4) is satisfied with the probability β. Since the point values of the probabilities P_{A_i}, $i=1, 2$, belong, with the confidence probability β, to the appropriate confidence intervals

$$P_{A_1} \in \langle P^c_{A_1}, r_{A_1} \rangle, \quad P_{A_2} \in \langle P^c_{A_2}, r_{A_2} \rangle, \qquad 1-P_{A_1} \in \langle 1-P^c_{A_1}, r_{A_1} \rangle \text{ and } 1-P_{A_2} \in \langle 1-P^c_{A_2}, r_{A_2} \rangle,$$

the following inequalities are true:

$$P^c_{A_1} - r_{A_1} \le P_{A_1} \le P^c_{A_1} + r_{A_1}, \tag{33}$$

$$P^c_{A_2} - r_{A_2} \le P_{A_2} \le P^c_{A_2} + r_{A_2}, \tag{34}$$

$$1-P^c_{A_1} - r_{A_1} \le 1-P_{A_1} \le 1-P^c_{A_1} + r_{A_1}, \tag{35}$$

$$1-P^c_{A_2} - r_{A_2} \le 1-P_{A_2} \le 1-P^c_{A_2} + r_{A_2}. \tag{36}$$

Form inequalities (34) and (35) it follows that

$$(1-P_{A_1})P_{A_2} \ge (1-P^c_{A_1} - r_{A_1})(P^c_{A_2} - r_{A_2}), \tag{37}$$

and inequalities (33) and (36), in view of $\lambda > 0$, yield

$$\lambda (1-P_{A_2})P_{A_1} \le \lambda (P^c_{A_1} + r_{A_1})(1-P^c_{A_2} + r_{A_2}). \tag{38}$$

Thus, if condition (29) holds, then based on (37) and (38) we have

$$(1-P_{A_1})P_{A_2} \ge (1-P^c_{A_1} - r_{A_1})(P^c_{A_2} - r_{A_2}) > \lambda (P^c_{A_1} + r_{A_1})(1-P^c_{A_2} + r_{A_2}) \ge \lambda (1-P_{A_2})P_{A_1}.$$

Therefore, if in the conflict situation S_{12} a decision is made in favor of V_1 when condition (29) holds, then condition (4), which provides, according to Lemma 1, the minimum average probability of the error of collective decision, also holds with the probability β.

The validity of other conditions of Theorem 1 can be proved similarly. The theorem is proved.

It is obvious that a conflict remains unsettled if the intervals $\mathbf{L} = \langle L^c, r_L \rangle$ and $\mathbf{W} = \langle W^c, r_W \rangle$ intersect; therefore, none of the conditions of Theorem 1 is fulfilled. Note that the intervals $\mathbf{L} = \langle L^c, r_L \rangle$ and $\mathbf{W} = \langle W^c, r_W \rangle$ may intersect even when the confidence intervals I_{A_1} and I_{A_2} do not intersect.

Figure 1 shows the domains *1* and *2* of the collective decision according to conditions (29) and (30) when the plant states are estimated with the a priori probabilities $P(V_1) = 0.8$ and $P(V_2) = 0.2$ in the conflict situation S_{12}. The grey color denotes the ranges of the frequencies $P^*_{A_1}$ (the experts' qualifications) for which the conflict situation cannot be settled since the intervals $\mathbf{L} = \langle L^c, r_L \rangle$ and $\mathbf{W} = \langle W^c, r_W \rangle$ intersect. It is easy to see that the domain of unsettled conflict decreases with increase in the size n of the experimental sampling and increases with increase in the confidence probability β.

Of practical interest are the conditions imposed on the size n of the experimental sample from which the experts' qualifications are estimated for the subsequent settlement of conflict situations according to the model proposed.

Before proving the theorem that specifies these conditions, let us prove an auxiliary lemma.

LEMMA 2. The functions

$$f_1(n) = P^c_X(n) - r_X(n),$$

$$f_2(n) = 1 - P^c_X(n) - r_X(n)$$

strictly monotonically increase with *n*, and the functions

$$f_3(n) = P^c_X(n) + r_X(n),$$

$$f_4(n) = 1 - P^c_X(n) + r_X(n)$$

strictly monotonically decrease with increase in *n*.

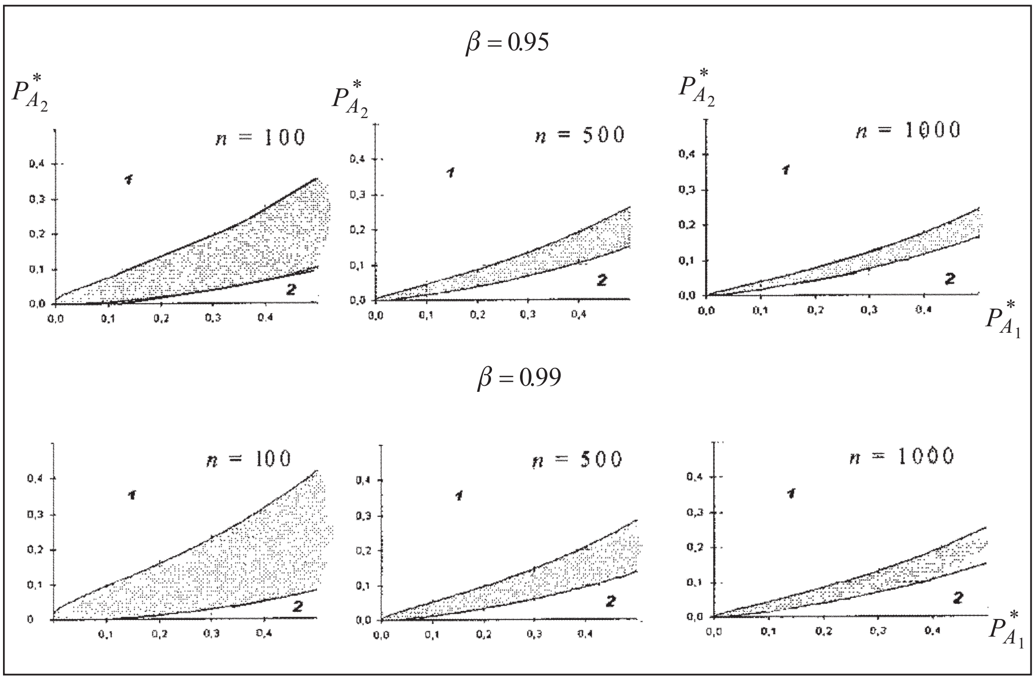

Fig. 1. The domains of settling a conflict S_{12} for $\lambda = 0.25$: a collective decision in favor of V_1 (1) and V_2 (2).

Proof. Let us consider $f_1(n)$ as a function of a continuous argument n and show that the first derivative of this function is positive for any $n > 0$, i.e.,

$$f_1'(n) = (P_X^c - r_X)' = \frac{t_\beta^2}{2n^2}\left(\frac{P_X^*(1-P_X^*) + \frac{t_\beta^2}{2n}}{t_\beta\sqrt{\frac{P_X^*(1-P_X^*)}{n} + \frac{t_\beta^2}{4n^2}}} - 1\right) > 0.$$

Since $\frac{t_\beta^2}{2n^2}$ is certainly positive, it will suffice to show that

$$\frac{P_X^*(1-P_X^*) + \frac{t_\beta^2}{2n}}{t_\beta\sqrt{\frac{P_X^*(1-P_X^*)}{n} + \frac{t_\beta^2}{4n^2}}} - 1 > 0$$

or

$$P_X^*(1-P_X^*) + \frac{t_\beta^2}{2n} > t_\beta\sqrt{\frac{P_X^*(1-P_X^*)}{n} + \frac{t_\beta^2}{4n^2}}.$$

Since both parts of the last inequality are positive, it is equivalent to the inequality

$$\left(P_X^*(1-P_X^*) + \frac{t_\beta^2}{2n}\right)^2 > \left(t_\beta\sqrt{\frac{P_X^*(1-P_X^*)}{n} + \frac{t_\beta^2}{4n^2}}\right)^2,$$

which after elementary transformations takes the form

$$(P_X^*(1-P_X^*))^2 + \frac{t_\beta^2}{n}P_X^*(1-P_X^*) + \frac{t_\beta^2}{4n^2} > \frac{t_\beta^2}{n}P_X^*(1-P_X^*) + \frac{t_\beta^4}{4n^2}.$$

As a result, we obtain the inequality $(P_X^* (1-P_X^*))^2 > 0$ which proves that the function $f_1(n)$ strictly monotonically increases for any $n > 0$.

It can be shown in the same way that the function $f_2(n)$ strictly monotonically increases, and the functions $f_3(n)$, $f_4(n)$ strictly monotonically decrease for any continuous $n > 0$. It is obvious that the lemma remains true for integer (discrete) values of $n > 1$.

Lemma 2 is proved.

THEOREM 2. For any $P(V_i)$ and β, there exists $n^* > 0$ such that once the expert' qualifications (the error rates $P_{A_i}^*$) are estimated using a representative sample of the size $n > n^*$, the optimal model of collective decisions provides settling possible conflict situations based on the conditions of Theorem 1.

Proof. To prove the theorem, it will suffice to show that for any $P_{A_i}^*$, irrespective of $P(V_i)$ and β, settling the conflict situations S_{12} and S_{21} is always possible by increasing the number of experiments to narrow down the confidence intervals (12).

Let us consider an example of settling the conflict situation S_{12}. Let for some $n = n_1 > 0$ there exist estimates $P_{A_i}^*$ such that for given $P(V_i)$ and β the intervals **L**, **W** intersect, i.e.,

$$L^c(n_1) - r_L(n_1) < W^c(n_1) + r_W(n_1)$$

or

$$L^c(n_1) + r_L(n_1) > W^c(n_1) - r_W(n_1).$$

It is clearly impossible to settle the conflict situation S_{12} in this case since neither (29) nor (30) is fulfilled.

Since, according to Lemma 2, the functions $P_{A_2}^c(n) - r_{A_2}(n)$ and $1 - P_{A_1}^c(n) - r_{A_1}(n)$ strictly monotonically increase with n, the function

$$L^c(n) - r_L(n) = \left(P_{A_2}^c(n) - r_{A_2}(n)\right)\left(1 - P_{A_1}^c(n) - r_{A_1}(n)\right)$$

increases too.

On the other hand, according to Lemma 2, the functions $P_{A_2}^c(n) + r_{A_2}(n)$ and $1 - P_{A_1}^c(n) + r_{A_1}(n)$ strictly monotonically decrease with increase in n. Therefore, the function

$$W^c(n) + r_W(n) = \left(P_{A_2}^c(n) + r_{A_2}(n)\right)\left(1 - P_{A_1}^c(n) + r_{A_1}(n)\right)$$

also strictly monotonically decreases with increase in n.

It follows herefrom that there always exists $n^* > n_1$ such that

$$L^c(n^*) - r_L(n^*) = W^c(n^*) + r_W(n^*). \tag{39}$$

Similar reasoning suggests that there always exists $n^* > n_1$ such that

$$L^c(n^*) + r_L(n^*) = W^c(n^*) - r_W(n^*), \tag{40}$$

and thus either condition (29) or (30) is satisfied for $n > n^*$.

The possibility of settling the conflict situation S_{21} can be proved similarly. Theorem 2 is proved.

A practically important corollary of Theorem 2 is the possibility to numerically estimate the size n of the experimental sample, which is necessary for settling conflict situations. For the given $P(V_i)$ and β, the necessary number of experiments n_0 for which conflict situations can be settled based on relations (29)–(32) depends on the experts' qualifications and is determined by the relation

$$n_0 = \left[t_\beta^2 \frac{\sqrt{\lambda}\,(P_{A_2}^* + \sqrt{\lambda}\,P_{A_1}^*)(\sqrt{\lambda}\,(1-P_{A_2}^*) + (1-P_{A_1}^*))}{(P_{A_2}^*(1-P_{A_1}^*) - \lambda P_{A_1}^*(1-P_{A_2}^*))^2} + 1 \right],$$

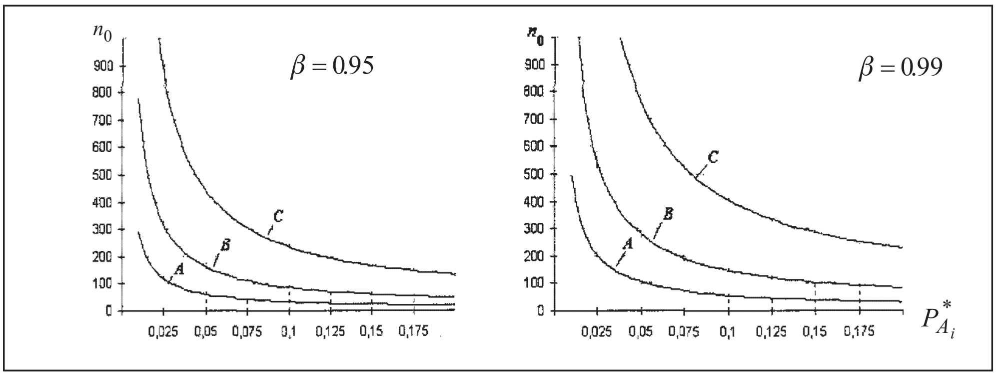

Fig. 2. Dependences of the required size of the experimental sample on experts' qualifications for $\lambda = 0.11$ and $\lambda = 9$ (A), $\lambda = 0.25$ and $\lambda = 4$ (B), and $\lambda = 0.43$ and $\lambda = 2.33$ (C).

where $[\eta]$ is the integer part of η. The number n_0 results from algebraic transforms of the expression for the greatest root of Eqs. (39) and (40).

Figure 2 presents the number of experiments n_0 as a function of the experts' error rates $P^* = P^*_{A_1} = P^*_{A_2}$ for fixed λ and confidence probabilities $\beta = 0.95$ and $\beta = 0.99$.

Note that the necessary number of experiments n_0 increases as the experts' qualifications increases (the frequencies $P^*_{A_i}$ decrease) and also as $\lambda \to 1$.

It is worthy of note that a formal model of collective decision-making, based on point values of the probabilities of experts' errors, is much simpler in the case of collective estimate of the plant state with equal a priori probabilities. Indeed, as is seen from (4)–(7), the optimal collective decision in conflict situations always coincides with the decision of the most qualified expert if $\lambda = 1$. This fact may underlie a simplified interval model of collective decision-making.

THEOREM 3. Let two experts, A_1 and A_2, make independent decisions on the current state of the plant $V_j \in \{V_1, V_2\}$ with equal a priori probabilities, i.e., $P(V_1) = P(V_2)$. Then if the frequencies of erratic decisions $P^*_{A_i}(n)$ of each expert are preestimated using the sample of n observations, then, with the probability β, the collective decision $D = D(\delta_1, \delta_2)$ minimizes average error probability if

1) in the conflict situation S_{12} a decision is made in favor of V_1 when $P^c_{A_2} - r_{A_2} > P^c_{A_1} + r_{A_1}$, and in favor of V_2 when $P^c_{A_2} + r_{A_2} < P^c_{A_1} - r_{A_1}$;

2) in the conflict situation S_{21}, a decision is made in favor of V_1 when $P^c_{A_1} - r_{A_1} > P^c_{A_2} + r_{A_2}$, and in favor of V_2 when $P^c_{A_1} + r_{A_1} < P^c_{A_2} - r_{A_2}$.

The proof of the theorem is similar to that of Theorem 2.

Figure 3 shows the domains *1* and *2* that provide optimal collective decisions in conflict situations according to Theorem 3. As in the above case, the domains of unsettled conflicts between two partial decisions of experts (grey color in Fig. 3) narrow down as the size n of the sample increases and the confidence probability β decreases, approaching the diagonal $P_{A_2} = P_{A_1}$.

For practical implementation of the model, according to Theorem 3, it is required to determine the error rates $P^*_{A_i}$ from a sample of the necessary size n_0, which can be derived from the equations

$$P^*_{A_2} \pm t_\beta \sqrt{\frac{P^*_{A_2}(1-P^*_{A_2})}{n} + \frac{t^2_\beta}{4n^2}} = P^*_{A_1} \mp t_\beta \sqrt{\frac{P^*_{A_1}(1-P^*_{A_1})}{n} + \frac{t^2_\beta}{4n^2}}$$

for given confidence probability β.

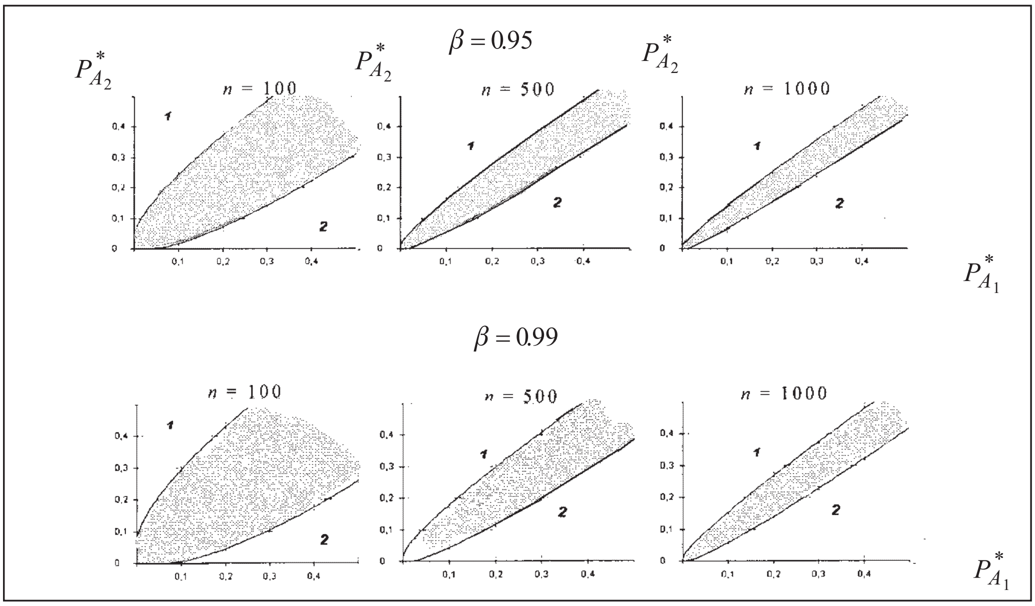

Fig. 3. The domains of optimal collective decisions for $\lambda = 1$ in favor of the states V_1 (1) and V_2 (2).

POSSIBLE GENERALIZATIONS

The interval model of collective decision-making in conflict situations proposed here can be naturally generalized to the case where not a priori probabilities $P(V_j)$ of states of the plant but their estimates (frequencies) $P^*_{V_j}$ obtained from an experimental sample are known.

Natural generalization of the results obtained is also the model of collective decision-making $D(\delta_1, \ldots, \delta_N)$ when a current state of the plant with $M \geq 2$ possible states $V_1, \ldots, V_M$ is estimated based on partial decisions $\delta_1, \ldots, \delta_N$ of a group of groups $N \geq 2$ independent experts, with the preestimated rates

$$P^*_{im} = \frac{E_{im}}{n_m}, \quad i = 1, \ldots, N, \quad m = 1, \ldots, M,$$

of errors E_{im} made by the ith expert in n_m experiments, in which the plant certainly was in the mth state. In this case, the generalized interval model of settling conflict situations that minimizes, with the probability β, the average probability of the error of a collective decision, is reduced to the following scheme.

Let S be a combination of partial decisions $\delta_1, \ldots, \delta_N$ of the experts $A_1, \ldots, A_N$, expressed as indicator functions

$$\delta_i = m \text{ if } A_i \text{ decides for the } m\text{th state, } m \in [1, \ldots, M].$$

Denote by I_m the set of the numbers of experts who decided in favor of the mth state of the plant ($m = 1, \ldots, M$). It is obvious that $I_l \cap I_j = \varnothing$ for any $l, j \in [1, M]$ and $I_1 \cup \ldots \cup I_M = \{1, \ldots, M\}$.

Arguing as in the proof of Theorem 1, it is easy to verify that the collective inference rule to minimize, with the confidence probability β, the average error probability in conflict situations is as follows: a decision is made in favor of the kth state if for any $m = 1, \ldots, M$, $m \neq k$,

$$(P^c_{V_k} - r_{V_k}) \prod_{i \in I_k} (1 - P^c_{A_{ik}} - r_{A_{ik}}) \prod_{i \notin I_k} (P^c_{A_{ik}} - r_{A_{ik}})$$
$$> (P^c_{V_m} + r_{V_m}) \prod_{i \in I_m} (1 - P^c_{A_{im}} + r_{A_{im}}) \prod_{i \notin I_m} (P^c_{A_{im}} + r_{A_{im}}).$$

Here $P^c_{V_m}$, r_{V_m} are, respectively, the centers and radii of the confidence intervals $I_{V_m}(\beta, n)$ of the a priori probability of the mth state of the plant, and $P^c_{A_{im}}$ and $r_{A_{im}}$ are, respectively, the centers and radii of the confidence intervals

$I_{A_{im}}(\beta, n)$ of the conditional probabilities of the errors of the *i*th expert for the *m*th state of the plant, depending on the confidence probability β and the size n of the experimental sample.

The further possible generalization of such interval model is the formal characterization of the conditions providing optimal decision-making in conflict situations with respect to the minimum mean risk. In this case, it is also expedient to specify the elements of the payoff matrix not as point quantities but as intervals with known boundaries.

Thus, we have developed formal models of collective decision-making. They are based on the Bayesian mechanisms of settling conflicts, which do not require the probabilities of experts' errors to be known a priori, in contrast to the well-known models. The models assume that experts' error rates are estimated from a finite sample of observations with known states of the plant and then possible conflict situations are settled based on this a priori knowledge. It has been shown that the model minimizes, with the probability β, the average error probability of a collective decision (Theorem 1). The necessary size of the experimental sample for the model of collective decisions has been established (Theorem 2). A simplified interval model of settling conflict situations has also been proposed, which proves a collective decision in favor of one of the two possible states of the plant with equal a priori probabilities (Theorem 3).

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